ON THE FORMS OF THE STRESS-STRAIN RELATION FOR INITIALLY ISOTROPIC NONELASTIC BODIES (GEOMETRIC ASPECT OF THE QUESTION)

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The theory of tensors depending on one scalar argument can be called the theory of tensor curves. Below, we will give an account of some of the properties of such curves with a view to applying the results to the problems of continuous media (in particular, to the question of the relation between stresses and deformations in nonelastic solids). The treatment will be confined to only three-dimensional symmetric tensors of second order. For the sake of simplifying the formulas, the components of the tensors are given in orthogonal Cartesian systems of coordinates.

1. Three-dimensional, symmetric tensor of second degree as an element of a six-dimensional space. Let us consider the manifold of three-dimensional symmetric tensors of second order

$$T = (T_{ij}) = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}, \qquad T_{ij} = T_{ji} \tag{1.1}$$

It forms a six-dimensional, linear, metric space in which the scalar product of its elements A and B is defined by the equation

$$(AB) = A_{ij}B_{ij} \tag{1.2}$$

The norm is formed from the scalar product, i.e.

$$\| A \| = \sqrt{A_{ij}A_{ij}} \tag{1.3}$$

and the distance between the points is equal to

$$\rho(AB) = \sqrt{(A_{ij} - B_{ij})(A_{ij} - B_{ij})} \tag{1.4}$$

Apart from this, the elements of H_6 are subject to the condition

$$A_{ik}B_{kj} + B_{ik}A_{kj} \in H_6 \tag{1.5}$$

which follows from the rule for multiplication of tensors (and coincides with the rule for multiplication of matrices).

If we did not have this last requirement, we could have enumerated the components of an arbitrary tensor in any order, e.g.

$$T_{11} = T_1, \quad T_{22} = T_2, \quad T_{33} = T_3, \quad \sqrt{2}T_{12} = T_4$$
$$\sqrt{2}T_{13} = T_5, \quad \sqrt{2}\, T_{23} = T_6 \tag{1.6}$$

and then treat the T_j as components of a vector in a six-dimensional Euclidean space. However, condition (1.5) excludes this possibility and compels us to treat H_6 as a special type of space (the space of symmetric tensors of second order), the basic properties of which will be dealt with in the first four sections of the paper.

That H_6 is six-dimensional follows from the fact that any three-dimensional symmetric tensor of the second order can be represented in the form (see, for example, [1])

$$T_{ij} = \sum_{m=1}^{6} t_{(m)}\, h_{ij}^{(m)} \tag{1.7}$$

where $h_{ij}^{(m)}$ are six arbitrarily chosen, linearly independent, three-dimensional tensors of the second order, and $t_{(m)}$ are six invariant coefficients. We will call expression (1.7) the expansion of T_{ij} in the tensor basis $h_{ij}^{(m)}$, and $t_{(m)}$ the components of T_{ij} along the tensors of this basis.

The determination of $t_{(m)}$ for given T_{ij} and $h_{ij}^{(m)}$ leads, in general, to the solution of a system of six linear, algebraic equations with six unknowns. However, this problem is greatly simplified if the basis $h_{ij}^{(m)}$ is orthogonal, i.e. if

$$h_{ij}^{(m)} h_{ij}^{(n)} = \delta_{mn} = \begin{cases} 1 & (m = n) \\ 0 & (m \neq n) \end{cases} \tag{1.8}$$

In this case the $t_{(m)}$ can be determined directly from the formulas

$$t_{(m)} = T_{ij}h_{ij}^{(m)} \tag{1.9}$$

It is obvious that any six symmetric, mutually orthogonal tensors of second degree are linearly independent, i.e. they form a basis. In fact, by multiplying the expression

$$\sum_{m=1}^{6} \alpha_{(m)} h_{ij}^{(m)} = 0 \tag{1.10}$$

scalarly by all the $h_{ij}^{(m)}$ (consecutively), we obtain $\alpha_{(m)} = 0$. It is also clear that there is no non-zero tensor orthogonal to all the basic tensors. This follows from (1.9), according to which all the components of a tensor that is orthogonal to the basic tensors are zero. Henceforth, we will assume that the basic tensors have been orthonormalized (provided no mention is made to the contrary).

Let us consider, along with $h_{ij}^{(m)}$, another basis $h_{ij}^{\vee(m)}$, and represent its tensors in the form of an expansion in terms of the tensors of the first basis (and *vice versa*)

$$h^{\vee}{}_{ij}^{(n)} = \sum_{m=1}^{6} \lambda_{(nm)} h_{ij}^{(m)}, \qquad h_{ij}^{(n)} = \sum_{m=1}^{6} \lambda_{(mn)} h^{\vee}{}_{ij}^{(m)} \tag{1.11}$$

From (1.8) and its analogous expression

$$h^{\vee}{}_{ij}^{(m)} \ h_{ij}^{\vee(n)} = \delta_{mn} \tag{1.12}$$

it follows that

$$\sum_{k=1}^{6} \lambda_{(mk)} \lambda_{(kn)} = \sum_{k=1}^{6} \lambda_{(km)} \lambda_{(kn)} = \delta_{mn} \tag{1.13}$$

i.e. the coefficients $\lambda_{(mn)}$ have the same properties as the cosines of the angles between the axes of two mutually orthogonal Cartesian coordinate systems in a six-dimensional Euclidean space.

Since

$$h_{ij}^{(m)} \delta_{ij} = h_{ii}^{(m)} = (h_m) \tag{1.14}$$

the expansion of a unit tensor in terms of tensors of an arbitrary orthonormal basis has the form

$$\delta_{ij} = \sum_{m=1}^{6} (h_m) h_{ij}^{(m)} \tag{1.15}$$

Multiplying this identity scalarly by δ_{ij}, we find

$$\sum_{m=1}^{6} (h_m)^2 = 3 \tag{1.16}$$

This relation connects the linear invariants of the tensors of an orthonormal basis and enables us to express one of them in terms of the other five.

We raise the first of the identities (1.11) to the second power (in the tensor sense). Then we will have

$$\check{h}_{ik}^{(m)}\, \check{h}_{kj}^{(m)} = \sum_{p=1}^{6} \sum_{q=1}^{6} \lambda_{(mp)}\, \lambda_{(mq)}\, h_{ik}^{(p)}\, h_{kj}^{(q)} \tag{1.17}$$

Summing (1.17) over all m and taking (1.13) into account, we obtain

$$\sum_{m=1}^{6} h_{ik}^{(m)}\, h_{kj}^{(m)} = \sum_{m=1}^{6} \check{h}_{ik}^{(m)}\, \check{h}_{kj}^{(m)} = C_{ij} \tag{1.18}$$

where C_{ij} is a constant symmetric tensor of second order, equal for all orthonormal tensor bases. In view of the isotropic nature of a three-dimensional Euclidean space (in it, there are no preferred directions), the present tensor can be none other than isotropic, i.e.

$$C_{ij} = C\, \delta_{ij} \tag{1.19}$$

By substituting (1.19) into (1.18) and contracting the indices i, j (on account of the fact that the basic tensors have been normed), we find that the invariant coefficient is $C = 2$. Thus, finally

$$\sum_{m=1}^{6} h_{ik}^{(m)}\, h_{kj}^{(m)} = 2\delta_{ij} \tag{1.20}$$

Formulas (1.20) and (1.16) express the basic tensor and basic vector properties of orthonormal tensor bases. By making use of them, it is possible to express any one of the basic tensors in terms of the remaining five and the unit tensor δ_{ij}.

In fact, let five orthonormal tensors $h_{ij}^{(m)}$ ($m = 1, 2, 3, 4, 5$) be prescribed. Then, by virtue of (1.20) one can write

$$h_{ik}\, h_{kj} = \Phi_{ij} \tag{1.21}$$

where $h_{ij} = h_{ij}^{(6)}$ is the required sixth basic tensor, and

$$\Phi_{ij} = 2\delta_{ij} - \sum_{m=1}^{5} h_{ik}^{(m)}\, h_{kj}^{(m)}, \qquad \Phi_{ii} = (\Phi) = 1 \tag{1.22}$$

is a known vector. We raise identity (1.21) to the second power

$$h_{ik}h_{kp}h_{pq}h_{qj} = \Phi_{ik}\Phi_{kj} \tag{1.23}$$

and, in order to transform this, use can be made of the Hamilton-Cayley theorem, according to which (see, for example, [2, p.108])

$$h_{ik}h_{kp}h_{pq}h_{qj} = (h)\left[\tfrac{1}{6}(h)^3 - \tfrac{1}{2}(h) + \tfrac{1}{3}(h^3)\right]\delta_{ij} + \tfrac{1}{3}\left[(h^3) - (h)^3\right]h_{ij} + \tfrac{1}{2}\left[1 + (h)^2\right]h_{ik}h_{kj} \qquad (1.24)$$

where

$$(h) = h_{ii}, \qquad (h^2) = h_{ij}h_{ij} = 1, \qquad (h^3) = h_{ik}h_{kj}h_{ij} \qquad (1.25)$$

By substituting (1.24) into (1.23) and calculating (1.21), we find

$$\left[(h^3) - (h)^3\right]h_{ij} = 3\Phi_{ik}\Phi_{kj} - \tfrac{3}{2}\left[1 + (h)^2\right]\Phi_{ij} - (h)\left[\tfrac{1}{2}(h)^3 - \tfrac{3}{2}(h) + (h^3)\right]\delta_{ij} \qquad (1.26)$$

Here we have introduced the first invariant $h_{ii} = (h)$, which apart from sign, can be determined from equation (1.16). Similarly, with regard to the invariant (h^3); after multiplying (1.26) scalarly by δ_{ij}, we obtain the following expression in terms of known quantities:

$$(h^3) = \frac{3}{4}\frac{1}{(h)}\left[(\Phi^2) - \frac{1}{2} + (h)^2 - \frac{1}{6}(h)^4\right] \qquad (1.27)$$

By the same token, formula (1.26) can be regarded as the definition (to within the sign) of the tensor h_{ij} in terms of the five prescribed orthonormal tensors $h_{ij}^{(m)}$.

An exceptional case arises when, in accordance with (1.16), it turns out that $(h) = 0$. Then (1.26) assumes the form

$$(h^3)\,h_{ij} = 3\Phi_{ik}\Phi_{kj} - \tfrac{3}{2}\Phi_{ij} \qquad (1.28)$$

and one must argue somewhat differently.

By multiplying equations (1.21) and (1.28) together scalarly, we have

$$(h^3)^2 = 3(\Phi^3) - \tfrac{3}{2}(\Phi^2) = 3\left[(\Phi^3) - \tfrac{1}{4}\right] \qquad (1.29)$$

In the particular case when h_{ij} happens to be a deviator, by introducing the preceding expression for (h^3) into (1.29), we are led to the formula determining h_{ij} in terms of the given tensors

$$h_{ij} = \pm\sqrt{3}\,\frac{\Phi_{ik}\Phi_{kj} - {}^1/_2\Phi_{ij}}{\sqrt{(\Phi^3) - {}^1/_4}} \qquad (1.30)$$

It was proved in [3] that the invariants $D_{ij}D_{ij} = (D^2)$ and $D_{ik}D_{kj}D_{ij} = (D^3)$ of an arbitrary deviator D_{ij} are subject to the inequality

$$-1 \leqslant \sqrt{6}\,\frac{(D^3)}{(D^2)^{3/2}} \leqslant 1 \qquad (1.31)$$

Hence, since it follows from (1.26) that $(\Phi^2) = 1/2$ when $(h) = 0$, it is possible to infer that

$$(\Phi^3) - \frac{1}{4} \geqslant 0 \tag{1.32}$$

and, consequently, the radical occurring in (1.30) is always real. Thus, the problem of finding the normalized tensor orthogonal to five orthonormalized tensors always has a solution.

2. On the subspaces in H_6. Starting from an arbitrary, orthonormal basis $h_{ij}^{(m)}$, it is possible to select two mutually orthogonal subspaces H_n and H_{6-n} from H_6. To the first will be assigned all tensors representable in the form

$$T_{ij}^{(1)} = \sum_{k=1}^{n} t_{(k)}^{(1)} h_{ij}^{(k)} \tag{2.1}$$

and to the second all tensors representable in the form

$$T_{ij}^{(2)} = \sum_{k=n+1}^{6} t_{(k)}^{(2)} h_{ij}^{(k)} \tag{2.2}$$

The tensors $h_{ij}^{(m)}$ $(m \leqslant n)$ form an orthonormal basis in H_n, and the tensors $h_{ij}^{(m)}$ $(m \geqslant n+1)$ form an orthonormal basis in H_{6-n}.

The transformation of one of the orthonormal bases into the other in H_n and H_{6-n} can be effected by the formulas

$$\check{h}_{ij}^{(m)} = \sum_{k=1}^{n} \lambda_{(mk)}^{(1)} h_{ij}^{(k)}, \qquad \check{h}_{ij}^{(m)} = \sum_{k=n+1}^{6} \lambda_{(mk)}^{(2)} h_{ij}^{(k)} \tag{2.3}$$

where the coefficients $\lambda_{(mk)}^{(1)}$, $\lambda_{mk}^{(2)}$ are subject to the equalities

$$\begin{gathered}\sum_{k=1}^{n} \lambda_{(pk)}^{(1)} \lambda_{(qk)}^{(1)} = \sum_{k=1}^{n} \lambda_{(kp)}^{(1)} \lambda_{(kq)}^{(1)} = \delta_{pq} \\ \sum_{k=n+1}^{6} \lambda_{(pk)}^{(2)} \lambda_{(qk)}^{(2)} = \sum_{k=n+1}^{6} \lambda_{(kp)}^{(2)} \lambda_{(kq)}^{(2)} = \delta_{pq}\end{gathered} \tag{2.4}$$

Of course, it is possible to divide H_6 into a larger number of mutually orthogonal subspaces. Moreover, an arbitrary element of H_6 can be represented in the form of a sum of elements belonging to mutually orthogonal subspaces, all of which are six-dimensional. Let us examine the properties of certain subspaces.

2.1. The deviatoric subspace D_5. We choose the coefficients λ_{6m} of the transformation in (1.11) as

$$\lambda_{6m} = \frac{1}{\sqrt{3}} (h_m) \tag{2.5}$$

Then, according to (1.15) and (1.16)

$$h_{ij}^{\vee\,(6)} = h_{ij}^{\vee} = \frac{1}{\sqrt{3}}\,\delta_{ij} = \delta_{ij}^{(*)} \tag{2.6}$$

i.e. the sixth tensor of the basis $h_{ij}^{\vee\,(m)}$ will be the normalized, isotropic tensor

$$\delta_{ij}^{(*)}\,\delta_{ij}^{(*)} = 1, \qquad \delta_{ii}^{(*)} = \sqrt{3} \tag{2.7}$$

The remaining five tensors $d_{ij}^{(m)} = h_{ij}^{\vee\,(m)}$ ($m = 1, 2, 3, 4, 5$), in view of the conditions

$$d_{ij}^{(m)}\,\delta_{ij}^{(*)} = d_{ij}^{(m)}\,\delta_{ij} = d_{ii}^{(m)} = 0$$

turn out to be deviators.

By making use of the tensors $d_{ij}^{(m)}$, δ_{ij}, it is possible to split H_6 into two mutually orthogonal subspaces: the one-dimensional one $\delta_{ij}^{(*)}$ (the space of isotropic tensors) and the five-dimensional one D_5, all the tensors of which are deviators.

Hence, it follows that an arbitrary deviator can be represented in the form

$$D_{ij} = \sum_{m=1}^{5} D_{(m)}\,d_{ij}^{(m)} \tag{2.8}$$

where $d_{ij}^{(m)}$ are five orthonormal deviators. The six-dimensional basis $\delta_{ij}^{(*)}$, $d_{ij}^{(m)}$ is subject to condition (1.20) and, consequently

$$\sum_{m=1}^{5} d_{ik}^{(m)}\,d_{kj}^{(m)} = \frac{5}{3}\,\delta_{ij} \tag{2.9}$$

This formula allows us to express any one of the five orthonormal deviators (for example, $d_{ij}^{(5)} = d_{ij}$) in terms of the other four and δ_{ij}. The corresponding calculations have already been given at the end of the preceding section. They lead to formula (1.30), in which, for the present case, Φ_{ij} must be understood to mean

$$\Phi_{ij} = \frac{5}{3}\,\delta_{ij} - \sum_{m=1}^{4} d_{ik}^{(m)}\,d_{kj}^{(m)} \tag{2.10}$$

It should be emphasized that the five-dimensional space D_5 of deviators has nothing in common with the five-dimensional vector space considered in the works [4,5] and others.

In the cited works on the Cartesian axes of a five-dimensional space certain linear combinations of the components of tensors in a Cartesian system of coordinates of a three-dimensional space were set aside. These linear combinations can then be interpreted as the components of five-dimensional vectors. However, the latter leads to the fact that the operations in a five-dimensional space lose invariant character and, as a rule, have meaning only when definite systems of coordinates are simultaneously fixed in both the three- and five-dimensional spaces. The latter is also obvious from the formulation of Theorem 1 in [4]. For details, see [6-9].

2.2. The subspace of deviators having a common principal direction. Tensors $\Omega_{ij} \in H_6$ are called axisymmetric, if two of their principal values are equal. If such a tensor is fully determined by only one of its principal directions, this will be called its axis.

If a normal deviator is axisymmetric, it will be subject to the condition

$$\Omega_{ij} = \pm \sqrt{6}\left[\Omega_{ik}\,\Omega_{kj} - \tfrac{1}{3}\delta_{ij}\right] \tag{2.11}$$

where the upper sign relates to the case when Ω_{ij} is a deviator of the tensile type, and the lower sign when Ω_{ij} is a compressive type. Formula (2.11) can be verified by carrying it over to the principal axes of Ω_{ij} and taking into consideration that

$$\Omega_1 = \pm\sqrt{\tfrac{2}{3}}, \qquad \Omega_2 = \Omega_3 = \mp\frac{1}{\sqrt{6}} \tag{2.12}$$

Henceforth, for the sake of definiteness, the symbol Ω_{ij} will denote a normalized deviator of the compressive type. It follows from (2.11) that for such deviators

$$(\Omega^3) = \Omega_{ik}\,\Omega_{kj}\,\Omega_{ij} = \frac{1}{\sqrt{6}} \tag{2.13}$$

If c_{ij} is an arbitrary normalized deviator, it can be connected with Ω_{ij}, the axis of which coincides with one of the principal directions of c_{ij}, by the formula

$$\Omega_{ij} = \frac{1}{\sqrt{6}\,(c^2 - {}^1/_6)}\left[cc_{ij} + c_{ik}\,c_{kj} - \tfrac{1}{3}\delta_{ij}\right] \tag{2.14}$$

where c is the principal value of c_{ij} corresponding to the axis of Ω_{ij}. Equation (2.14) can be verified by referring it to the principal axes of c_{ij}.

It follows from (2.14) that, if two normed deviators $c_{ij}^{(1)}$ and $c_{ij}^{(2)}$ have one common principal direction, the following relation will subsist between them:

$$\frac{1}{c_1^2 - 1/6}\left[c_1 c_{ij}^{(1)} + c_{ik}^{(1)} c_{kj}^{(1)} - \frac{1}{3}\delta_{ij}\right] = \frac{1}{c_2^2 - 1/6}\left[c_2 c_{ij}^{(2)} + c_{ik}^{(2)} c_{kj}^{(2)} - \frac{1}{3}\delta_{ij}\right] \qquad (2.15)$$

where c_1 and c_2 are the principal values of $c_{ij}^{(1)}$ and $c_{ij}^{(2)}$ corresponding to their common principal direction.

We will elucidate the conditions of orthonormality of the deviators $c_{ij}^{(1)}$ and $c_{ij}^{(2)}$. The table of direction cosines between their principal axes has the form:

	$c_1^{(2)}$	$c_2^{(2)}$	$c_3^{(2)} = c_2$
$c_1^{(1)}$	$\cos\gamma_{12}$	$\sin\gamma_{12}$	0
$c_2^{(2)}$	$-\sin\gamma_{12}$	$\cos\gamma_{12}$	0
$c_3^{(1)} = c_1$	0	0	1

Here, γ_{12} is the angle between the principal axes $c_1^{(1)}$ and $c_1^{(2)}$. Hence, the components of the tensor $c_{ij}^{(2)}$ in the principal axes of $c_{ij}^{(2)}$ can be expressed by the formulas

$$c_{11}^{(2)} = c_1^{(2)} \cos^2\gamma_{12} + c_2^{(2)} \sin^2\gamma_{12} \qquad (2.16)$$

$$c_{22}^{(2)} = c_1^{(2)} \sin^2\gamma_{12} + c_2^{(2)} \cos^2\gamma_{12} \qquad (2.17)$$
$$c_{33}^{(2)} = c_3^{(2)} = c_2$$

By subjecting (2.17) to the requirement $c_{ij}^{(1)}\ c_{ij}^{(2)} = 0$, we are led to the formula

$$\cos 2\gamma_{12} = -\alpha_1\alpha_2 \qquad \left(\alpha_k = \sqrt{3}\,\frac{c_1^{(k)} + c_2^{(k)}}{c_1^{(k)} - c_2^{(k)}}\right) \qquad (2.18)$$

If $c_{ij}^{(1)}$ and $c_{ij}^{(2)}$ are normalized, then

$$\alpha_k = -\sqrt{3}\,\frac{c_3^{(k)}}{\sqrt{2-3\,(c_3^{(k)})^2}} = -\sqrt{3}\,\frac{c_k}{\sqrt{2-3c_k^2}} \qquad (2.19)$$

As is obvious from (2.18), there is an infinity of normalized deviators $c_{ij}^{(2)}$ which have the one principal direction in common with the given normalized deviator $c_{ij}^{(1)}$ and which are orthogonal to it. We will consider an arbitrary pair of such tensors and try to select yet another normalized deviator, which is orthogonal to them and has with them a common principal direction. Then, besides (2.18) we will also have two analogous equations

$$\cos 2\gamma_{13} = -\alpha_1\alpha_3,\ \cos 2\gamma_{23} = -\alpha_2\alpha_3 \qquad (\gamma_{23} = \gamma_{12} - \gamma_{13}) \qquad (2.20)$$

By making use of (2.18) and (2.20), we obtain

$$\alpha_1^2\alpha_2^2 + \alpha_1^2\alpha_3^2 + \alpha_2^2\alpha_3^2 = 1 - 2\,\alpha_1^2\alpha_2^2\alpha_3^2 \qquad (2.21)$$

Hence

$$\alpha_3 = \pm \sqrt{\frac{1-\alpha_1^2\alpha_2^2}{\alpha_1^2+\alpha_2^2+2\alpha_1^2\alpha_2^2}} \tag{2.22}$$

From (2.18) and (2.19), it follows that a change of sign in front of α_3 is equivalent to a change of sign in front of $c_{ij}^{(3)}$. By the same token, it turns out that to each pair of orthonormal deviators $c_{ij}^{(1)}$, $c_{ij}^{(2)}$ having a common principal direction there corresponds a unique (to within sign) deviator $c_{ij}^{(3)}$ which is orthogonal to them both. From the account, it also follows that in an arbitrary deviatoric basis $d_{ij}^{(m)}$, no more than three of its tensors can have a common principal direction.

Concerning the manifold $D_3^{(*)}$ of deviators having a common principal direction with $\Omega_{ij}^{(*)}$, the following statements are valid:

a) an arbitrary linear combination of such deviators $c_{ij} \in D_3^*$,

b) any four deviators, having a common principal direction are linearly dependent,

c) an arbitrary $c_{ij} \in D_3^*$ is representable in the form

$$c_{ij} = \sum_{m=1}^{3} c_{(m)}\, c_{ij}^{(m)} \tag{2.23}$$

where $c_{ij}^{(m)}$ are three orthonormal deviators from D_3^*,

d) from (a), (b) and (c) it follows that each manifold D_3^* is a three-dimensional subspace of D_5 in H_6,

e) to every axisymmetric deviator Ω_{ij} a certain D_3^* can be associated,

f) to its elements will belong all deviators having a common principal axis which coincides with the axis of $\Omega^*_{ij}(\Omega^*)$,

g) an arbitrary symmetric tensor of second order is formed by means of some linear combination of the elements of D_3^*, since, for example

$$c_{ik}^{(1)} c_{kj}^{(1)}, \quad c_{ik}^{(1)} c_{kj}^{(2)} + c_{ik}^{(2)} c_{kj}^{(1)}, \quad c_{ik}^{(1)} c_{kp}^{(1)} c_{pj}^{(2)} + c_{ik}^{(2)} c_{kp}^{(1)} c_{pj}^{(1)}, \ldots \tag{2.24}$$

has principal direction coinciding with that of Ω^*.

The transformation of one orthonormal basis $c_{ij}^{(m)}$ into another $\check{c}_{ij}^{(m)}$ is accomplished by means of the formula

$$\check{c}_{ij}^{(m)} = \sum_{n=1}^{3} \lambda_{(mn)}\, c_{ij}^{(n)} \qquad \left(\sum_{k=1}^{3} \lambda_{mk}\lambda_{nk} = \sum_{k=1}^{3} \lambda_{km}\lambda_{kn} = \delta_{mn}\right) \tag{2.25}$$

From (2.25), it follows that

$$\sum_{m=1}^{3} c_{ik}^{(m)} c_{kj}^{(m)} = \sum_{m=1}^{3} \check{c}_{ik}^{(m)} \check{c}_{kj}^{(m)} = A_{ij} \tag{2.26}$$

where A_{ij} is a tensor, which remains invariant under arbitrary transformations (2.25) into the corresponding $D_3{}^*$.

By means of elementary calculations it is possible to establish that every shear deviator S_{ij} (i.e. a tensor with the principal values $S_1 = -S_2$, $S_3 = 0$) is orthogonal to an axisymmetric tensor Ω_{ij}, if the axis of the latter is perpendicular to the plane of shear. On the other hand, by virtue of (2.18), two shear deviators $S_{ij}^{(1)}$, $S_{ij}^{(2)}$ (with the same shear plane) are orthogonal, if the angle between their principal directions $\gamma_{12} = \pi/4$. Hence, the simplest class of orthogonal deviators, having a principal direction in common with Ω^*, is $\Omega_{ij}{}^*$ and two mutually orthogonal deviators of the shear type $S_{ij}^{(1)}$, $S_{ij}^{(2)}$ the plane of shear of which is perpendicular to Ω^*. By making use of this most simple basis, it is possible to determine A_{ij}.

In this respect it should be noted that, according to (2.14)

$$S_{ik}^{(1)} S_{kj}^{(1)} - \tfrac{1}{3}\delta_{ij} = S_{ik}^{(2)} S_{kj}^{(2)} - \tfrac{1}{3}\delta_{ij} = -\frac{1}{\sqrt{6}}\Omega_{ij}^* \tag{2.27}$$

By setting in (2.27)

$$c_{ij}^{(1)} = \Omega_{ij}{}^*, \quad c_{ij}^{(2)} = S_{ij}^{(1)}, \quad c_{ij}^{(3)} = S_{ij}^{(2)} \tag{2.28}$$

we find

$$A_{ij} = \delta_{ij} - \frac{1}{\sqrt{6}}\Omega_{ij}{}^* \tag{2.29}$$

We represent $\Omega^*{}_{ij}$ in the form of an expansion in terms of the basic deviators $c_{ij}^{(m)}$

$$\Omega_{ij}^* = \sum_{m=1}^{3} \Omega_{(m)} c_{ij}^{(m)}, \qquad \Omega_{(m)} = \Omega_{ij}^* \, c_{ij}^{(m)} \tag{2.30}$$

However, in agreement with (2.14)

$$\sqrt{6}\,\Omega_{(m)} = \frac{1}{c_m^2 - {}^1/_6}\left[c_m + (c_m^3)\right] \tag{2.31}$$

On the basis of the Hamilton-Cayley theorem

$$c_m^3 = \tfrac{1}{2} c_m + \tfrac{1}{3}(c_m^3) \tag{2.32}$$

Hence

$$\Omega_{(m)} = \sqrt{{}^3/_2}\, c_m \tag{2.33}$$

By the same token

$$\Omega_{ij}^{*} = \sqrt{\tfrac{3}{2}} \sum_{m=1}^{3} c_m\, c_{ij}^{(m)} \tag{2.34}$$

A consequence of the present formula is the equation

$$\sum_{m=1}^{3} c_m^2 = \tfrac{2}{3} \tag{2.35}$$

By substituting (2.34) into (2.29), and then (2.29) into (2.26), we obtain the equation

$$\sum_{m=1}^{3} \left[c_{ik}^{(m)}\, c_{kj}^{(m)} + \tfrac{1}{2} c_m\, c_{ij}^{(m)} \right] = \delta_{ij} \tag{2.36}$$

connecting the three orthonormal deviators having a common principal direction. This equation (together with (2.35)) enables us to express one of these deviators in terms of the two others and δ_{ij} (to within the sign).

However, we will not dwell on this, in as far as the analogous expressions were proved in Section 1.

2.3. Subspace of coaxial deviators. If two deviators a_{ij}, b_{ij} are coaxial, then, as is well known, they are connected by the equation

$$b_{ij} = A_1 a_{ij} + A_2 \left[a_{ik}\, a_{kj} - \tfrac{1}{3} (a^2) \delta_{ij} \right] \tag{2.37}$$

where A_1, A_2 are scalar coefficients which can be expressed in terms of the invariants of the deviators of a_{ij}, b_{ij} (see, for example, [3]).

By taking into account that a_{ij} and b_{ij} are normalized and mutually orthogonal, we obtain

$$b_{ij} = \pm \left[\tan\gamma\, a_{ij} - \sqrt{6} \sec\gamma\, (a_{ik}\, a_{kj} - \tfrac{1}{3} \delta_{ij}) \right] \tag{2.38}$$

where

$$\sin\gamma = \sqrt{6}\, (a^3) = \sqrt{6}\, a_{ik}\, a_{kj}\, a_{ij} \tag{2.39}$$

We note that inequality (1.32) is valid for an arbitrary deviator and, consequently, for a normalized deviator

$$-1 \leqslant \sqrt{6}\, (a^3) \leqslant 1 \tag{2.40}$$

By the same token, to every normalized deviator a_{ij} there corresponds a unique (to within sign), coaxial, normal, normalized deviator b_{ij}. An exception case arises when

$$\sin\gamma = \pm 1, \qquad (a^3) = \pm \frac{1}{\sqrt{6}} \tag{2.41}$$

In addition, a_{ij} can turn out to be an axisymmetric deviator and be subject to equality (2.11), in view of which the right-hand side of (2.33) becomes indeterminate of the form 0/0. This indeterminacy cannot be removed. The fact is that in this case there is an infinity of deviators coaxial with and orthogonal to a_{ij}.

They will all be deviators of the shearing type in which the shear plane is perpendicular to the axis of a_{ij} (see 2.2). However, an arbitrary pair of such deviators will not be coaxial to each other. Therefore, even though in the present case there is freedom of choice of the normalized deviator coaxial and normal to a_{ij}, the assertion that there are no third mutually orthogonal, normalized, coaxial deviators however remains in force.

With respect to the set $D_2{}^*$ of coaxial deviators we have the following assertions:

1) An arbitrary linear combination of such vectors $A_{ij} \in D_2{}^*$.

2) An arbitrary multiplicative tensor, which has been formed from such deviators, is a coaxial element of $D_2{}^*$.

3) Each $D_2{}^*$ is a two-dimensional subspace in H_6 and D_5, and the present $D_2{}^*$ belongs to such a subspace in an arbitrary $D_3{}^*$, in which Ω^* corresponds to one of the principal directions of the deviators.

3. On the number of tensors sufficient for the formulation of a basis in H_6. A basis in H_6 consists of six linearly independent, symmetric tensors of second degree. However, a nonlinear dependence between the base tensors is not excluded. In this connection, the question arises as to the smallest number of tensors sufficient for the formation of the basis. In passing, we will also touch on the question of the formation of tensor bases from vectors.

3.1. Dyadic bases. We will consider the set Λ of three-dimensional symmetric tensors of second order, having the form

$$g_{ij} = \partial_i e_j + \partial_j e_i \tag{3.1}$$

where ∂_i, and e_i are two arbitrary three-dimensional vectors. The invariants g_{ij} can be expressed in terms of the invariant vectors ∂_i, and e_i in the following manner:

$$g_{ii} = (g) = 2\partial e \cos\varphi, \qquad g_{ij}\, g_{ij} = (g^2) = 2\partial^2 e^2 (1 + \cos^2\varphi) \tag{3.2}$$
$$g_{ik}\, g_{kj}\, g_{ij} = (g^3) = 2\partial^3 e^3 (3 + \cos^2\varphi)\cos\varphi$$

where

$$\textrm{э} = \sqrt{\textrm{э}_i \textrm{э}_i}, \qquad e = \sqrt{e_i e_i}, \qquad \cos\varphi = \frac{\textrm{э}_i e_i}{\textrm{э} e} \tag{3.3}$$

From (3.2) it follows that in the form (3.1) only such symmetric tensors of second order can be represented, in which the invariants are subject to the relation

$$2(g^3) = (g)[3(g^2) - (g)^2] \tag{3.4}$$

By setting here $(g) = 0$, we obtain $(g^3) = 0$, i.e. from the whole set of deviators, it is only the tensors of the shear type that can be represented in the given form (3.1). Isotropic tensors cannot be expressed in the form (3.1).

The set Λ, being a particular set of elements occurring in the space H_6, is not one of its subspaces because $g_{ij}^{(1)} + g_{ij}^{(2)}$ is not an element of Λ in general.

By taking three mutually orthogonal unit vectors $\textrm{э}^{(1)}$, $\textrm{э}^{(2)}$, $\textrm{э}^{(3)}$, it is possible to form from them six symmetric tensors of second order

$$g_{ij}^{(m)} = \textrm{э}_i^{(m)}\textrm{э}_j^{(m)}, \quad g_{ij}^{(3+m)} = \frac{1}{\sqrt{2}}[\textrm{э}_i^{(p)}\textrm{э}_j^{(q)} + \textrm{э}_i^{(q)}\textrm{э}_j^{(p)}] \tag{3.5}$$

where m, p, q assume the values 1, 2, 3, and also $m \neq p \neq q$.

By referring the tensors (3.5) to axes with the directions of $\textrm{э}^{(1)}$, $\textrm{э}^{(2)}$ and $\textrm{э}^{(3)}$, it is easily established that

$$g_{ij}^{(k)} g_{ij}^{(l)} = \delta_{kl} \tag{3.6}$$

i.e. the tensors (3.5) form an orthonormal dyadic basis in H_6. The components of an arbitrary $T \in H_6$ with respect to the tensors of such a basis are equal to

$$t_{(m)} = T_{mm}, \qquad t_{3+m} = \sqrt{2}T_{pq} \tag{3.7}$$

where T_{ij} are components of the tensor T in Cartesian coordinates, the axes of which coincide with $\textrm{э}^{(1)}$, $\textrm{э}^{(2)}$ and $\textrm{э}^{(3)}$. If $\textrm{э}^{(m)}$ are not orthogonal, then we lose the mutual orthogonality of $g_{ij}^{(k)}$ in (3.5). Nevertheless, in this case also they remain linearly independent (under the condition that $\textrm{э}^{(m)}$ are not coplanar) and can be used as six basic tensors.

In [1], tensors of arbitrary order are successively represented in the form of expansions in multiplicative tensors which have been formed from the basic vectors (in particular, in dyads if attention is directed to second order tensors), and also use is made of both orthogonal and nonorthogonal bases. This representation is very convenient for differentiated tensors referred to a fixed system of coordinates.

3.2. Bases formed from multiplicative tensors of the second order. It is obvious that it is impossible to construct a basis in H_6 having in all only one symmetric tensor a_{ij}. Of course, from this tensor it is possible to construct a unique, symmetric tensor $a_{ik}a_{kj}$, which is generally linearly independent of it. Every other multiplicative tensor, as, for example, $a_{ik}a_{kp}a_{pj}$, $a_{ik}a_{kp}a_{pq}a_{qj}$, ..., in view of the Hamilton-Cayley formula, can be expressed in terms of δ_{ij}, a_{ij}, $a_{ik}a_{kj}$. Therefore, the number of tensors sufficient for the formation of a basis cannot be more than two.

We will elucidate the sufficiency of this number.

Let a_{ij}, b_{ij} be two linearly independent tensors. It is possible to construct from them the following six symmetric tensors:

$$a_{ik}a_{kj}, \quad b_{ik}b_{kj}, \quad a_{ik}b_{kj} + b_{ik}a_{kj}, \quad a_{ik}a_{kp}b_{pj} + b_{ik}a_{kp}a_{pj}$$
$$b_{ik}b_{kp}a_{pj} + a_{ik}b_{kp}b_{pj}, \qquad a_{ik}a_{kp}b_{pq}b_{qj} + b_{ik}b_{kp}a_{pq}a_{q} \tag{3.8}$$

All other multiplicative, symmetric tensors of the second order will be expressible in terms of the nine tensors δ_{ij}, a_{ij}, b_{ij} and (3.8) on the basis of the Hamilton-Cayley theorem generalized to the case of two tensors [10]. And what is more, it is possible to assert that between the above-mentioned nine tensors there are always n linear relations, where n is not less than 3 because the above nine tensors are elements of H_6.

It is clear that when $n > 3$ it is not possible to form a basis from these nine tensors, because among them there would not be six linearly independent ones. It is not difficult to quote examples corresponding to this case. In particular, if a_{ij} and b_{ij} have a common principal direction Ω^*, then all symmetric tensors formed from them will have this very same principal direction. They will belong to the four-dimensional subspace $D_3{}^*$, δ_{ij}, and, consequently, each five of them will be linearly dependent (Section 2).

By the same token, even if one of the principal directions of a_{ij} and b_{ij} should coincide, we cannot construct a tensor basis from them. And what is more, this is impossible even if three such tensors are given. If, however, a_{ij} and b_{ij} do not have a common principal direction, then, as was proved in [11], the coefficients in the expression

$$T_{ij} = \alpha_0\delta_{ij} + \alpha_1 a_{ij} + \alpha_2 b_{ij} + \alpha_3 a_{ik}a_{kj} + \alpha_4 b_{ik}b_{kj} +$$
$$+ \alpha_5 (a_{ik}b_{kj} + b_{ik}a_{kj}) + \alpha_6 (a_{ik}a_{kp}b_{pj} + b_{ik}a_{kp}a_{pj}) +$$
$$+ \alpha_7 (b_{ik}b_{kp}a_{pj} + a_{ik}b_{kp}b_{pj}) \tag{3.9}$$

can all be chosen so as to satisfy equation (3.9) for an arbitrary tensor T_{ij}.

The last statement is equivalent to the assertion that, in the case considered, among the nine tensors δ_{ij}, a_{ij}, b_{ij} (3.8) there are six linearly independent ones, from which can be formed a basis in H_6.

From the foregoing we draw the two conclusions:

1) For the construction of a basis in H_6 it is sufficient to have three three-dimensional vectors, provided they are not coplanar.

2) For the construction of a basis in H_6 it is sufficient to have two three-dimensional symmetric tensors of second order, provided they do not have a common principal direction.

4. Generalization of Serret-Frenet formulas. We will consider the tensor $R_{ij} \in H_6$ which is a function of the single scalar parameter λ.

The derivative

$$\frac{dR_{ij}}{d\lambda} = r_{ij} \tag{4.1}$$

will not be a normalized tensor in general. However, if we change over to the new argument

$$s = \int_0^\lambda \sqrt{\frac{dR_{ij}}{d\lambda}\frac{dR_{ij}}{d\lambda}}\, d\lambda \tag{4.2}$$

then the tensor

$$\frac{dR_{ij}}{ds} = r_{ij}^{(1)} \tag{4.3}$$

will prove to be normalized.

We will call argument s chosen in the above manner the length of the tensor curve R_{ij}, and $r_{ij}^{(1)}$ will denote the normalized tangent tensor to it. Now we form the following sequence of recurrence relations:

$$\frac{dr_{ij}^{(1)}}{ds} = \varkappa_1 r_{ij}^{(2)}, \qquad \frac{dr_{ij}^{(2)}}{ds} = -\varkappa_1 r_{ij}^{(1)} + \varkappa_2 r_{ij}^{(3)}$$

$$\frac{dr_{ij}^{(3)}}{ds} = -\varkappa_2 r_{ij}^{(2)} + \varkappa_3 r_{ij}^{(4)}, \ldots, \quad \frac{dr_{ij}^{(k)}}{ds} = -\varkappa_{k-1} r_{ij}^{(k-1)} + \varkappa_k r_{ij}^{(k+1)}, \ldots \tag{4.4}$$

In each of these there appears one tensor and one scalar coefficient that were not in the preceding formulas. By the same token, formulas (4.4) will determine the tensors $r_{ij}^{(k)}$ $(k \geqslant 2)$ in terms of the tensor

$r_{ij}^{(1)}$ prescribed by formula (4.3).

As far as the scalar coefficients κ_k are concerned, they will be chosen so that all the successive tensors are normalized. In this connection, we will prove that

$$r_{ij}^{(m)} r_{ij}^{(n)} = \delta_{mn} \tag{4.5}$$

i.e. that $r_{ij}^{(m)}$ are not only normalized but also mutually orthogonal.

In fact, by multiplying the first of formulas (4.4) scalarly by $r_{ij}^{(1)}$, we obtain

$$r_{ij}^{(1)} \frac{dr_{ij}^{(1)}}{ds} = \frac{1}{2} \frac{d}{ds} (r_{ij}^{(1)} r_{ij}^{(1)}) = 0 = \varkappa_1 r_{ij}^{(1)} r_{ij}^{(2)} \tag{4.6}$$

Hence, if $\kappa_1 \neq 0$, we have

$$r_{ij}^{(1)} r_{ij}^{(2)} = 0 \tag{4.7}$$

Now we multiply the second of formulas (4.4) scalarly by $r_{ij}^{(1)}$, and also by $r_{ij}^{(2)}$. Then we will have

$$r_{ij}^{(1)} \frac{dr_{ij}^{(2)}}{ds} = \frac{d}{ds} (r_{ij}^{(1)} r_{ij}^{(2)}) - r_{ij}^{(2)} \frac{dr_{ij}^{(1)}}{ds} = -\varkappa_1 = -\varkappa_1 + \varkappa_2 r_{ij}^{(1)} r_{ij}^{(3)} \tag{4.8}$$

$$r_{ij}^{(2)} \frac{dr_{ij}^{(2)}}{ds} = \frac{1}{2} \frac{d}{ds} (r_{ij}^{(2)} r_{ij}^{(2)}) = 0 = \varkappa_2 r_{ij}^{(2)} r_{ij}^{(3)} \tag{4.9}$$

Hence, if $\kappa_2 \neq 0$, we have

$$r_{ij}^{(1)} r_{ij}^{(3)} = r_{ij}^{(2)} r_{ij}^{(3)} = 0 \tag{4.10}$$

By completely analogous arguments, we are led to the conclusion that all the tensors in (4.4) are mutually orthogonal. But then the given sequence of formulas cannot be continued indefinitely; from the fact that there are no more than six mutually orthogonal, symmetric tensors of the second order, it follows that the sequence must terminate not later than the sixth formula, i.e. $\kappa_6 = 0$ (although, in certain cases, it can terminate earlier).

Thus, for an arbitrary tensor curve in H_6, we have the following relations:

$$\frac{dr_{ij}^{(1)}}{ds} = \varkappa_1 r_{ij}^{(2)}, \quad \frac{dr_{ij}^{(2)}}{ds} = -\varkappa_1 r_{ij}^{(1)} + \varkappa_2 r_{ij}^{(3)}, \quad \frac{dr_{ij}^{(3)}}{ds} = -\varkappa_2 r_{ij}^{(2)} + \varkappa_3 r_{ij}^{(4)}$$
$$\frac{dr_{ij}^{(4)}}{ds} = -\varkappa_3 r_{ij}^{(3)} + \varkappa_4 r_{ij}^{(5)}, \quad \frac{dr_{ij}^{(5)}}{ds} = -\varkappa_4 r_{ij}^{(4)} + \varkappa_5 r_{ij}^{(6)}, \quad \frac{dr_{ij}^{(6)}}{ds} = -\varkappa_5 r_{ij}^{(5)} \tag{4.11}$$

which constitute a generalization of the Serret-Frenet formulas. They permit the determination, for each tensor curve $R_{ij}(s)$, of a natural reference frame $r_{ij}^{(m)}$, i.e. the set of six orthonormal tensors at each point of the curve.

If the parameters

$$\varkappa_m = \varkappa_m(s) \qquad (m = 1, 2, 3, 4, 5) \tag{4.12}$$

of a curvilinear tensor curve are prescribed, then (4.11) becomes a system of six linear, ordinary differential equations, the unknowns in which are the tensors $r_{ij}^{(m)}$. Their general solution can be written down in the form

$$r_{ij}^{(m)} = \sum_{n=1}^{6} C_{ij}^{(n)} f_{(mn)} \tag{4.13}$$

where $C_{ij}^{(n)}$ are six constant tensors playing the role of constants of integration, and the thirty six scalar functions $f_{(mn)}$ of the s argument are arbitrarily chosen in the form of a particular solution of the following six systems of equations:

$$\begin{aligned} &\frac{df_{(1n)}}{ds} = \varkappa_1 f_{(2n)}, \qquad \frac{df_{(2n)}}{ds} = -\varkappa_1 f_{(1n)} + \varkappa_2 f_{(3n)} \\ &\frac{df_{(3n)}}{ds} = -\varkappa_2 f_{(2n)} + \varkappa_3 f_{(4n)}, \qquad \frac{df_{(4n)}}{ds} = -\varkappa_3 f_{(3n)} + \varkappa_4 f_{(5n)} \\ &\frac{df_{(5n)}}{ds} = -\varkappa_4 f_{(4n)} + \varkappa_5 f_{(6n)}, \qquad \frac{df_{(6n)}}{ds} = -\varkappa_5 f_{(5n)} \end{aligned} \tag{4.14}$$

If the particular solutions $f_{(mn)}$ of system (4.14) are chosen so as to satisfy the initial conditions

$$f_{(mn)} = \delta_{mn} \quad \text{when } s = 0 \tag{4.15}$$

then it turns out that

$$C_{ij}^{(m)} = r_{ij}^{(m)}(0) \tag{4.16}$$

By the same token, when the $f_{(mn)}$ are chosen as above, the tensor constants $C_{ij}^{(m)}$ will form an orthogonal basis coinciding with the "natural reference frame" of the tensor curve at the point $s = 0$. Formulas (4.13) in the present case can be treated as the transformation of this constant basis into the "natural reference frame", formed at an arbitrary point s.

In agreement with the above choice of initial values for $f_{(mn)}$, we have

$$\sum_{k=1}^{6} f_{(mk)} f_{(nk)} = \sum_{k=1}^{6} f_{(km)} f_{(kn)} = \delta_{mn} \tag{4.17}$$

As has already been mentioned, sequence (4.4) can terminate before the sixth formula.

For example, if $R_{ij}(s)$ is a deviator, all the $r_{ij}^{(m)}$ will also be deviators. Then sequence (4.4) must terminate no later than at the fifth formula, becuase there are no more than five orthonormal deviations (Section 2). Thus, for deviator curves (which will be of greatest importance later on) we always have $\kappa_5 = 0$, i.e. they are a particular form of five-dimensional tensor curves belonging to the subspace D_5. If $R_{ij}(s)$ is not only a deviator but also retains one of the principal directions Ω^* when it otherwise varies, then sequence (4.4) must terminate at the third formula (i.e. $\kappa_3 = 0$). This follows from the fact that there are no more than three orthonormal deviators having a common principal direction (Section 2). Consequently, tensor curves of this class will be particular forms of three-dimensional tensor curves. If all three principal directions of the deviator $R_{ij}(s)$ remain fixed during the variation of the deviator $R_{ij}(s)$, then (4.4) terminates at the second formula, since there are no more than two coaxial, orthonormal deviators. In this case $\kappa_2 = 0$, and $R_{ij}(s)$ will be a particular form of "plane" (two-dimensional) tensor curves.

The resemblance of the theory of tensor curves to the theory of vector curves is obvious. However, there are also some differences between these theories. For example, the Serret-Frenet formulas for three-dimensional vector curves in the general case do not have a general integral, whereas equations (4.11) are always amenable to quadrature. Their general integral is expressed by

$$\sum_{m=1}^{6} r_{ik}^{(m)} r_{kj}^{(m)} = A_{ij} = \text{const} \tag{4.18}$$

which can be obtained by tensor multiplication of equations (4.11) by $r_{ij}^{(1)}, r_{ij}^{(2)}, \ldots, r_{ij}^{(6)}$, respectively, and by then summing all the equations. The necessity of the existence of the integral (4.18) is a consequence of (1.20).

5. Some applications to the question of the connection between stresses and deformations in nonelastic solids. In physics, problems often arise in connection with the establishment of relations between vector or tensor curves. The classical example is Newton's second law of mechanics, which can be treated as an isotropic connection between the vector curve $\mathbf{r}(t)$ (the trajectory of a material point) and another vector curve $\mathbf{F}(t)$ (the force).

The form of laws of this type is always subject to limitations of a geometric character, which are called forth by the finite dimensionality of the space. Thus, for example, independent of any sort of physical considerations, it is possible to assert that the force vector is representable in the form

$$\mathbf{F} = F_1\boldsymbol{\tau} + F_2\boldsymbol{\nu} + F_3\boldsymbol{\beta} \tag{5.1}$$

where $\boldsymbol{\tau}$, $\boldsymbol{\nu}$ and $\boldsymbol{\beta}$ are the unit vectors of the tangent, the normal and the binormal to the trajectory, and F_1, F_2 and F_3 are the projections of the force onto them.

Expression (5.1) can be transformed into the form

$$\mathbf{F} = \frac{1}{v}\left(F_1 - F_2\frac{\dot{v}}{\sqrt{w^2-\dot{v}^2}}\right)\dot{\mathbf{r}} + \frac{F_2}{\sqrt{w^2-\dot{v}^2}}\ddot{\mathbf{r}} + F_3\frac{\dot{\mathbf{r}}\times\ddot{\mathbf{r}}}{v\sqrt{w^2-\dot{v}^2}} \tag{5.2}$$

where

$$v = \sqrt{\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}},\ w = \sqrt{\ddot{\mathbf{r}}\cdot\ddot{\mathbf{r}}}$$

The necessity that the right-hand side of (5.2) remains invariant when $\mathbf{r}$ is changed into

$$\mathbf{R} = \mathbf{r} + \mathbf{v}_0 t \tag{5.3}$$

where $\mathbf{v}_0$ is a constant vector, i.e. by assuming the existence of invariant reference frames, we find that

$$\mathbf{F} = f\ddot{\mathbf{r}} \tag{5.4}$$

where f is an invariant which, in view of the adopted assumption, can be a function of any invariants of the derivatives of vector $\mathbf{r}$, apart from those containing $\dot{\mathbf{r}}$. If in addition it is demanded that the integral

$$\int_{r_0}^{r_1}\mathbf{F}\cdot d\mathbf{r} = \int_{r_0}^{r_1} f\ddot{\mathbf{r}}\cdot d\mathbf{r} = \frac{1}{2}\int_{r_0}^{r_1} f d(v^2) \tag{5.5}$$

should depend only on the values of the invariants of the motion at points r_0 and r_1, then it turns out that $f = m =$ const, and (5.4) becomes the well known formulation of the second law of mechanics. Then, (5.5) turns out to be the integral of the kinetic energy.

The above reasoning has two aspects: one is geometrical, the other physical. The first confines the search for possible forms of the relation between the vectors $\mathbf{r}$ and $\mathbf{F}$ to the trinomial formula (5.2), which thereby excludes from consideration all time derivatives of vector $\mathbf{r}$ of order greater than the second. The second aspect introduces two hypotheses, based on experience, which lead to the significant simplification of (5.2).

From this example it is clear that the role of the geometrical considerations in establishing the connection between vector curves (and, consequently, between tensor curves) is not to be exaggerated. A decisive role is obviously played by the considerations based on an analysis of the physics of the phenomena under consideration.

Nevertheless, certain benefit can also be derived from a preliminary investigation of the geometrical aspect of the problem.

We will consider the connection between the stress tensor σ_{ij} and the strain tensor ε_{ij} in a nonelastic solid, i.e. basically the question of the formulation of the constitutive equations for such solids. In contrast to the elastic body, in the present case there is no single-valued relation between σ_{ij} and ε_{ij} at each instant of time t. However (if the temperature distribution of the process of deformation is fixed by assuming, for example, that it is isotropic) it is possible to assert that the tensor curve $\sigma_{ij}(t)$ uniquely determines the tensor curve $\varepsilon_{ij}(t)$. In this connection, it will be assumed that at the initial instant $t = 0$ the body is undeformed ($\varepsilon_{ij}(0) = 0$) and free from stress ($\sigma_{ij}(0) = 0$). Moreover, we will consider the body to be initially isotropic, by assuming that in the constitutive equations of the material there appear no other tensors than σ_{ij}, ε_{ij} and their time derivatives.

We will represent ε_{ij} and σ_{ij} in the form

$$\begin{aligned}\varepsilon_{ij} &= \varepsilon_{ij}' + \tfrac{1}{3}e\delta_{ij} = E\partial_{ij} + \tfrac{1}{3}e\delta_{ij}\\ \sigma_{ij} &= \sigma_{ij}' + \tfrac{1}{3}\sigma\delta_{ij} = Ss_{ij} + \tfrac{1}{3}\sigma\delta_{ij}\end{aligned} \tag{5.6}$$

$$e = \varepsilon_{ii}, \qquad \sigma = \sigma_{ii}, \qquad E = \sqrt{\varepsilon_{ij}'\varepsilon_{ij}'}, \qquad S = \sqrt{\sigma_{ij}'\sigma_{ij}'} \tag{5.7}$$

where ε_{ij}' and σ_{ij}' are the deviators of the stress and strain tensors, and the normalized tensors ∂_{ij} and s_{ij} are determined by the formulas

$$\partial_{ij} = \frac{\varepsilon_{ij}'}{E}, \qquad s_{ij} = \frac{\sigma_{ij}'}{S} \tag{5.8}$$

Expressions (5.6) represent the tensor curves $\varepsilon_{ij}(t)$ and $\sigma_{ij}(t)$ in terms of their projections on the Cartesian space D_5 and the one-dimensional space δ_{ij}. Let the quantities $\partial_{ij}(t) = \partial_{ij}^{(1)}(t)$, $E(t)$ and $e(t)$ be prescribed, which, by virtue of (5.6) and (5.7), is equivalent to prescribing $\varepsilon_{ij}(t)$.

By following the preceding section, we can write

$$\begin{aligned}&\dot{\partial}_{ij}^{(1)} = \varkappa_1\partial_{ij}^{(2)}, \qquad \dot{\partial}_{ij}^{(2)} = -\varkappa_1\partial_{ij}^{(1)} + \varkappa_2\partial_{ij}^{(3)}\\ &\dot{\partial}_{ij}^{(3)} = -\varkappa_2\partial_{ij}^{(2)} + \varkappa_3\partial_{ij}^{(4)}, \qquad \dot{\partial}_{ij}^{(4)} = -\varkappa_3\partial_{ij}^{(3)} + \varkappa_4\partial_{ij}^{(5)}\\ &\dot{\partial}_{ij}^{(5)} = -\varkappa_4\partial_{ij}^{(4)}\end{aligned} \tag{5.9}$$

where $\partial_{ij}^{(m)}$ are five deviators which form an orthonormal basis in D_5 and are a "natural reference frame" for the tensor curve

$$R_{ij}(t) = \int_0^t \partial_{ij}^{(1)}\, dt \tag{5.10}$$

We will represent the stress deviator σ'_{ij} in the form of an expansion in the given reference frame

$$\sigma'_{ij} = \sum_{m=1}^{5} S_{(m)} \partial_{ij}^{(m)} \tag{5.11}$$

Since the body is regarded as initially isotropic, the coefficients in this expansion must be functions (or functionals) only of the invariants of the tensors ε_{ij} and σ_{ij} and of their time derivatives.

The deviator $\partial_{ij}^{(m)}$ can be expressed in terms of ε'_{ij} and the first of its four time derivatives. Therefore, (5.11) can be brought into the form

$$\sigma'_{ij} = \sum_{m=0}^{4} f_{(m)} \frac{d^m \varepsilon'_{ij}}{dt^m} \tag{5.12}$$

where $f_{(m)}$, as also $S_{(m)}$ are functions (or functionals) of the invariants of the tensors σ_{ij}, ε_{ij} and their derivatives.

Such, on the face of it, is the most general form of the relations between the stresses and deformations in an initially isotropic body. However, it is easy to quote examples which indicate the insufficient generality of (5.12). We will now consider the case of the one-dimensional deformation

$$\partial_{ij}^{(1)} = \partial_{ij}^{(0)} = \text{const}, \qquad \varepsilon'_{ij} = E(t)\, \partial_{ij}^{(0)} \tag{5.13}$$

This simplest deformation is characterized by a single deviator $\partial_{ij}^{(0)}$. However, from it we can in general construct yet another linearly independent deviator

$$\partial_{ik}^{(0)} \partial_{kj}^{(0)} - {}^1/_3 \delta_{ij}$$

Therefore, the tensor structure of the stress-deformation relations in the present case must have the form

$$\sigma'_{ij} = F_1 \partial_{ij}^{(0)} + F_2 [\partial_{ik}^{(0)} \partial_{kj}^{(0)} - {}^1/_3\, \delta_{ij}] = f_1 \varepsilon'_{ij} + f_2 [\varepsilon'_{ik} \varepsilon'_{kj} - {}^1/_3 E^2 \delta_{ij}] \tag{5.14}$$

where f_1 and f_2 depend on the invariants ε, e, their time derivatives and the third invariant of the deviator ε'_{ij}.

On the other hand, for this same case expression (5.12) gives in all only the linear equation

$$\sigma_{ij}' = f_1 \varepsilon_{ij}' \tag{5.15}$$

The essence of the contradiction between (5.14) and (5.15) consists in the fact that the former is quite general for the five dimensional curves $\varepsilon_{ij}'(t)$, in which all five deviators forming the "natural reference frame" have been completely determined. Formulas (5.12) lose generality when applied to the case of l-dimensional curves ($l < 5$) since, although in this connection there are only l deviators $\partial_{ij}^{(m)}$, we can, as a rule, construct from them yet another k deviators ($l + k \leqslant 5$), which are linearly independent of them and among themselves Therefore, from this fact that the curve ε_{ij} will be l-dimensional, it cannot at all follows that σ_{ij}' must also be l-dimensional, as occurs when (5.12) is adopted; the latter can also belong to a subspace with a larger number of dimensions.

If the curve $\varepsilon_{ij}'(t)$ is two-dimensional, then

$$\partial_{ij}^{(1)} = \partial_{ij}^{(10)} \cos\varphi + \partial_{ij}^{(20)} \sin\varphi, \qquad \partial_{ij}^{(2)} = -\partial_{ij}^{(10)} \sin\varphi + \partial_{ij}^{(20)} \cos\varphi \tag{5.16}$$
$$\varphi = \varphi(t), \qquad \varphi(0) = 0$$

and the process of deformation is characterized by two deviators $\partial_{ij}^{(10)}$, $\partial_{ij}^{(20)}$ determining a "plane" in D_5. Moreover, as is known from Section 3, the tensor structure of the relation between σ_{ij}' and ε_{ij}' will be dependent on the properties of these tensors and, indeed, if $\partial_{ij}^{(10)}$, $\partial_{ij}^{(20)}$ are coaxial, the relations in question are representable in the form

$$\sigma_{ij}' = f_1 \varepsilon_{ij}' + f_2 (\varepsilon_{ik}' \varepsilon_{kj}' - {}^1/_3 E^2 \delta_{ij}) \tag{5.17}$$

If $\partial_{ij}^{(10)}, \partial_{ij}^{(20)}$ have a common principal direction, then

$$\sigma_{ij}' = f_1 \varepsilon_{ij}' + f_2 \dot{\varepsilon}_{ij}' + f_3 \omega_{ij} \tag{5.18}$$

where ω_{ij} is a deviator linearly independent of ε_{ij}' and ε_{ij}', being a function of these two tensors. How it can be constructed has been indicated in 2.2. Finally, if ε_{ij}' and $\dot{\varepsilon}_{ij}'$ do not have a common principal direction, then, as follows from 3.2, the tensor structure of the curve can be written down in the form

$$\sigma_{ij}' = f_0 \delta_{ij} + f_1 \varepsilon_{ij}' + f_2 \dot{\varepsilon}_{ij}' + f_3 \varepsilon_{ik}' \varepsilon_{kj}' + f_4 \dot{\varepsilon}_{ik}' \varepsilon_{kj}' +$$
$$+ f_5 (\varepsilon_{ik}' \dot{\varepsilon}_{kj}' + \dot{\varepsilon}_{ik}' \varepsilon_{kj}') + f_6 (\varepsilon_{ik}' \varepsilon_{kp}' \dot{\varepsilon}_{pj}' + \dot{\varepsilon}_{ik}' \varepsilon_{kp}' \varepsilon_{pj}') +$$
$$+ f_7 (\dot{\varepsilon}_{ik}' \dot{\varepsilon}_{kp}' \varepsilon_{pj}' + \varepsilon_{ik}' \dot{\varepsilon}_{kp}' \dot{\varepsilon}_{pj}') \tag{5.19}$$

Thus, to a "two-dimensional" curve of deformations can correspond either the two-dimensional (5.17), or the three-dimensional (5.18) or, in the general case, the five-dimensional (5.19) stress deviator curve.

Special treatment must be given to those deformations during which one of the principal directions of ε_{ij}' remains fixed as the tensor itself varies. This case is of interest, because the investigation of the plastic behavior of materials is usually carried out on tubular specimens and fulfills the above-mentioned condition.

In this case, the curve of deformations $\varepsilon_{ij}'(t)$ turns out to be three-dimensional, i.e.

$$\begin{aligned} \vartheta_{ij}^{(1)} &= f_{11}\vartheta_{ij}^{(10)} + f_{12}\vartheta_{ij}^{(20)} + f_{13}\vartheta_{ij}^{(30)} \\ \vartheta_{ij}^{(2)} &= f_{21}\vartheta_{ij}^{(10)} + f_{22}\vartheta_{ij}^{(20)} + f_{23}\vartheta_{ij}^{(30)} \\ \vartheta_{ij}^{(3)} &= f_{31}\vartheta_{ij}^{(10)} + f_{32}\vartheta_{ij}^{(20)} + f_{33}\vartheta_{ij}^{(30)} \end{aligned} \qquad (5.20)$$

and the process of deformation is characterized by three deviators $\vartheta_{ij}^{(10)}$, $\vartheta_{ij}^{(20)}$ and $\vartheta_{ij}^{(30)}$ having a common principal direction. One of them, as has been shown in 2.2, can always be expressed in terms of two others, so that only two of the deviators in (5.20) are unknowns. Apart from this, it follows from 2.2 that from three deviators having a common principal direction we cannot construct another deviator that is linearly independent. In view of this, the tensor structure of $\sigma_{ij}(t)$ in the present case can be expressed by formula (5.18), i.e. the stress deviator, as well as the strain deviator, turns out to be a three-dimensional tensor curve.

Thus, the basic geometric properties of the stress-deformation relations in initially isotropic, nonelastic bodies can be represented in a lucid manner. It is doubtful whether further progress in this direction will be of value, since geometric considerations alone are clearly insufficient for the solution of the problem under investigation. Only by means of an analysis of the phenomena arising during the irreversible deformation of a solid, and the creation of corresponding physical models can lead to the establishment of laws describing deformations of this type. In actual fact, this is exactly how progress is made in the modern theory of plasticity and its basic direction – the theory of flow.

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